Communications in Partial Differential Equations, 33: 2208–2224, 2008

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300802523446

Multiplicity Result for a Scalar Field Equation on Compact Surfaces

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We consider a scalar field equation on compact surfaces which has variational structure. When the surface is a torus and a physical parameter ρ *belongs to* $(8\pi, 4\pi^2)$ *we show under some extra assumptions that, besides a local minimum, the functional admits at least other two saddle points.*

Keywords Geometric PDEs; Scalar field equations; Variational methods.

Mathematics Subject Classification 35J60; 35J50; 53A30.

1. Introduction

Let (Σ, g) be a compact Riemann surface without boundary and with volume equal to 1.

In this paper we consider the following mean field equation:

$$-\triangle_g u + \rho = \rho \frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} \quad x \in \Sigma, \;\; u \in H^1(\Sigma), \tag{1.1}$$

where $h \in C^\infty(\Sigma)$ is a positive function and ρ a real parameter.

This problem arises in statistical mechanics as a mean field equation for the Euler flow. More precisely, it has been proved in [5, 21] that, according to Onsager's vortex model, when the number of vortices is supposed to tend to $+\infty$, the stream function satisfies (1.1). In this interpretation the exponential is related to the Gibbs measure, which is finite provided $\rho > -8\pi$.

This PDE also concerns the description of self-dual condensates of some Chern–Simon–Higgs model; indeed via its solutions it is possible to describe the asymptotic behavior of a class of condensates (or multivortex) solutions which are relevant in theoretical physics and which were absent in the classical (Maxwell–Higgs) vortex theory (see [34, 35]).

Received March 05, 2008; Accepted August 20, 2008
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Besides, equation (1.1) is geometrically meaningful because it is related to the problem of prescribing the Gauss curvature, K_g, of a surface Σ via a conformal transformation of the metric. In fact, considering the conformal metric $\tilde{g} = e^{2w}g$ (with $\omega : \Sigma \to \mathbb{R}$ smooth), the Gauss curvature transforms by

$$-\triangle_g w + K_g = K_{\tilde{g}} e^{2w}.$$

In this context, of particular interest is the classical Uniformization Theorem, which asserts that every compact surface carries a conformal metric with constant curvature. Viceversa, given a surface with constant curvature one may ask whether it is possible to obtain conformal metrics for which the Gauss curvature becomes a given function. The latter is known as the Kazdan–Warner problem, or as the Nirenberg problem when Σ is the standard sphere (see for example [7, 8, 20]).

Problem (1.1) has a variational structure and solutions can be found as critical points of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \rho \int_\Sigma u\, dV_g - \rho \log \int_\Sigma h(x) e^u dV_g \quad u \in H^1(\Sigma). \tag{1.2}$$

Since equation (1.1) is invariant adding constants to u, we can restrict ourselves to the subspace $\overline{H}^1(\Sigma) \subset H^1(\Sigma)$ of the functions with zero average; so we will sometimes omit the second term in I_ρ.

When $\rho \leq 0$, existence and uniqueness of critical points for (1.2) follow easily. Hence we shall focus on the more interesting case $\rho > 0$. A consequence of the Moser–Trudinger inequality, see (2.1), is that, if $\rho \in (0, 8\pi)$, the functional I_ρ is bounded from below and coercive; thus solutions can be found as global minima.

When $\rho > 8\pi$, as I_ρ is unbounded both from above and below, solutions have to be found as saddle points. In [15] Ding, Jost, Li and Wang proved that, assuming $\rho \in (8\pi, 16\pi)$ and assuming that the genus of the surface is greater or equal to 1, there exists a solution to (1.1). In [24], Lin studied the case of the 2-sphere for $\rho \in (8\pi, 16\pi)$ by computing the Leray–Schauder degree for equation (1.1) (this degree does not depend on h and is always equal to -1). When $\rho \in (16\pi, 24\pi)$, Lin also proved that the degree on S^2 is equal to 0, so the author was not able to conclude the existence of a solution. More generally, Chen and Lin [12] computed the Leray–Schauder degree for (1.1) for any $\rho \neq 8\pi\mathbb{N}$. By means of this formula they deduced that, when the genus of Σ is positive, the degree is non zero and (1.1) admits a solution. In [28], Malchiodi presented a more direct proof and an interpretation of the degree formula obtained in [12]. Finally, when $\rho \neq 8k\pi$ ($k \in \mathbb{N}$), Djadli [16] generalized these previous results establishing the existence of a solution for any (Σ, g), [16]; to do that he deeply investigated the topology of low sublevels of I_ρ in order to perform a min-max scheme (already introduced in Djadli and Malchiodi [17]).

It is worth pointing out that is not known whether the Palais–Smale condition holds for the functional I_ρ, anyhow the problem can be bypassed using a method introduced by Struwe which exploits the monotonicity in ρ of $\frac{I_\rho}{\rho}$. In this way it is possible to get existence for almost every $\rho \in (8\pi, 16\pi)$. Then a crucial compactness property (Theorem 2.7), due to Li [22], which relies on some quantization results in [4] and [23], allows to obtain solutions for every $\rho \notin 8\pi\mathbb{N}$.

A variant of Struwe's method is due to Lucia [26] (see also Lemma 2.6 below), who presented a version of the classical Deformation lemma which holds true for I_ρ.

If ρ is an integer multiple of 8π the existence problem of (1.1) is much harder. Already the case $\rho = 8\pi$ is more delicate since I_ρ still has a lower bound but it is not coercive anymore (a far from complete list of references on the subject includes works by Chang and Yang [7], Chang, Gursky and Yang [8], Chen and Li [9], Nolasco and Tarantello [31], Ding, Jost, Li and Wang [14], Lucia [25] and references therein).

In the present paper we want to generalize a result by Struwe and Tarantello [33], dealing with the case of the flat torus T with $h \equiv 1$ and ρ varying in the range $(8\pi, 4\pi^2)$. In these hypotheses, $u = 0$ is clearly a solution of

$$-\triangle u + \rho = \rho \frac{e^u}{\int_T e^u dx}; \tag{1.3}$$

moreover because of the fact that, when $\rho \in (8\pi, 4\pi^2)$, $u = 0$ is a strict local minimum for I_ρ (see (3.1) below), the authors were able to exhibit a mountain-pass structure and to prove the existence of another solution of (1.3).

Theorem 1.1 ([33]). *Let Σ be the flat torus and let $h \equiv 1$. Then, for every $\rho \in (8\pi, 4\pi^2)$, there exists a non-trivial solution u_ρ of (1.3) satisfying $I_\rho(u_\rho) \geq (1 - \rho/4\pi^2)c_0$ for some constant $c_0 > 0$ independent of ρ.*

Perturbing h and g the functional I_ρ will still have a strict local minimum, $\bar{u}$, in a neighborhood of 0 and the same arguments allow to find a saddle point. We improve this result stating that, apart from $\bar{u}$, there are at least two critical points. Under the assumptions of Theorem 1.1 there are indeed infinitely-many non trivial solutions, arising from the translations in T of any given one. We show that indeed a multiplicity persists for generic data.

Theorem 1.2. *If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least two different saddle points.*

Remark 1.3. For the result of Theorem 1.2 to hold it is sufficient that the functional I_ρ possesses a strict local minimum.

In the above statement $S^{2\times 2}$ stands for the symmetric 2×2 matrices on T. To prove Theorem 1.2 we construct a new functional $\widetilde{I}_\rho$ which has the same critical points as I_ρ and which coincides with I_ρ away from the minimum, $\bar{u}$, where it takes very negative values. In such a way we enrich the topology of low sublevels of $\widetilde{I}_\rho$, which in particular result disconnected.

Next, by taking advantage of the topological properties of low and high sublevels of I_ρ, studied in [17, 29], we find two levels, $-L$ and b, such that the number of critical points of $\widetilde{I}_\rho$ in $\{-L \leq \widetilde{I}_\rho \leq b\}$ is at least two. To do this we use the notion of Lusternik–Schnirelmann relative category (roughly speaking a natural number measuring how a set is far from being contractible, when a subset is fixed); in particular, defining a projection of $\{\widetilde{I}_\rho \leq b\}$ in a homeomorphic image of the topological cone constructed on T, we are able to reduce the problem to the study of the category of a finite dimensional metric space. By our construction, the two critical points produced for $\widetilde{I}_\rho$ are also critical for $\widetilde{I}_\rho$, so the result follows.

We actually expect that a third (non trivial) solution might exist, see Remark 3.3 for more precise comments.

In Section 2 we collect some useful material concerning compactness properties for (1.1) and the topological structure of $\widetilde{I}_\rho$. In Section 3 we introduce the modified functional $\widetilde{I}_\rho$ and prove our multiplicity result.

2. Notation and Preliminaries

In this section we collect some facts needed in order to obtain the multiplicity result. First of all we consider some improvements of the Moser–Trudinger inequality which are useful to study the topological structure of the sublevels of I_ρ. Next, we state a deformation lemma, proved in [26], and a compactness property of solutions of (1.1) derived in [22]. These last results, for $\rho \neq 8k\pi$, allow us to overcome the possible failure of the (PS)-condition and to get a counterpart of the classical deformation lemma.

Let now fix our notation. The symbol $B_r(p)$ denotes the metric ball of radius r and center p, while $\operatorname{dist}(x, y)$ stands for the distance between two points $x, y \in \Sigma$. As already specified we set $\overline{H}^1(\Sigma) := \{u \in H^1(\Sigma) \mid \bar{u} := \int_\Sigma u = 0\}$, endowed with the norm $\|u\|_{\overline{H}^1(\Sigma)} := (\int_\Sigma |\nabla_g u|^2 dV_g)^{\frac{1}{2}}$.

Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula.

Finally, given a smooth functional $I : H^1(\Sigma) \to \mathbb{R}$ and a real number c, we set

$$I^c := \{u \in H^1(\Sigma) \mid I(u) \leq c\}$$
$$Z_c := \{u \in H^1(\Sigma) \mid DI(u) = 0 \text{ and } I(u) = c\}.$$

2.1. *The Moser–Trudinger Inequality and the Structure of Sublevels*

First of all we recall the well-known Moser–Trudinger inequality on compact surfaces.

Lemma 2.1 (Moser–Trudinger inequality). *There exists a constant C, depending only on (Σ, g) such that for all $u \in H^1(\Sigma)$*

$$\int_\Sigma e^{\frac{4\pi(u-\bar{u})^2}{\int_\Sigma |\nabla_g u|^2 dV_g}} \leq C. \tag{2.1}$$

As a consequence one has for all $u \in H^1(\Sigma)$

$$\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C. \tag{2.2}$$

Chen and Li [10] from this result showed that if e^u has integral controlled from below (in terms of $\int_\Sigma e^u dV_g$) into $(l+1)$ distinct regions of Σ, the constant $\frac{1}{16\pi}$ can be basically divided by $(l+1)$. Since we are interested in the behavior of the functional when $\rho \in (8\pi, 16\pi)$, it is sufficient to consider the case $l = 1$.

Lemma 2.2 [10]. *Let* Ω_1, Ω_2 *be subsets of* Σ *satisfying* $dist(\Omega_1, \Omega_2) \geq \delta_0$, *where* δ_0 *is a positive real number, and let* $\gamma_0 \in (0, \frac{1}{2})$. *Then, for any* $\tilde{\varepsilon} > 0$ *there exists a constant* $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ *such that*

$$\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq C + \frac{1}{32\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2 dV_g$$

for all the functions satisfying

$$\frac{\int_{\Omega_i} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \gamma_0, \quad i = 1, 2. \tag{2.3}$$

Qualitatively, when $\rho \in (8\pi, 16\pi)$ and (2.3) is satisfied, the functional I_ρ stays uniformly bounded from below. A consequence of this inequality is that, if $I_\rho(u)$ attains large negative values, e^u has to concentrate at one point of Σ. Indeed, using the previous Lemma and a covering argument, Ding, Jost, Li and Wang obtained the following result (see [16, 17] or [29]).

Lemma 2.3. *If* $\rho \in (8\pi, 16\pi)$, *the following property holds. For any* $\varepsilon > 0$ *and any* $r > 0$ *there exists a large positive* $L = L(\varepsilon, r)$ *such that, for every* $u \in H^1(\Sigma)$ *with* $I_\rho(u) \leq -L$, *there exists a point* $p \in \Sigma$ *such that*

$$\frac{\int_{\Sigma \setminus B_r(p)} e^u dV_g}{\int_\Sigma e^u dV_g} < \varepsilon. \tag{2.4}$$

Now we want to take advantage of the above improvement of Moser–Trudinger inequality to characterize the topology of low sublevels. We first point out that Σ can be mapped into very negative sublevels of I_ρ and that this map turns out to be non-trivial, in the sense that it carries some homology.

In order to do this, we need to define the functions $\tilde{\varphi}_{\lambda,x} : \Sigma \to \mathbb{R}$ by

$$\tilde{\varphi}_{\lambda,x}(y) = \log \left(\frac{\lambda}{1 + \lambda^2 \mathrm{d}_x{}^2(y)} \right)^2, \tag{2.5}$$

where λ is a positive real parameter and $\mathrm{d}_x(y) = \mathrm{d}_x(x, y)$, $x, y \in \Sigma$. Clearly, since the distance from a fixed point of Σ is a Lipschitz function, $\tilde{\varphi}_{\lambda,x}(y)$ is also Lipschitz in y and hence it belongs to $H^1(\Sigma)$.

As we will work in $\overline{H}^1(\Sigma)$, we will consider the normalized functions $\varphi_{\lambda,x} := \tilde{\varphi}_{\lambda,x} - \overline{\tilde{\varphi}_{\lambda,x}}$ and, for a fixed λ, the set $T_\lambda := \{\varphi_{\lambda,x} \mid x \in \Sigma\}$.

It is easy to see that (see e.g. [15]), as $\lambda \to +\infty$, $I_\rho(\varphi_{\lambda,x}) \to -\infty$ uniformly for $x \in \Sigma$; so once $L > 0$ is fixed, there exists λ large enough such that $\varphi_{\lambda,x} \in I_\rho^{-L}$ for any $x \in \Sigma$. In this way we get an immersion of Σ into arbitrarily low sublevels ($x \mapsto \varphi_{\lambda,x}$).

On the other hand, thanks to Lemma 2.3, we know that for $u \in I_\rho^{-L}$, $L \gg 0$, the probability measure $\frac{e^u dV_g}{\int_\Sigma e^u dV_g}$ is concentrated near some point of Σ. Roughly speaking, associating this point to u, we obtain a map from low sublevels into Σ

$$\Psi : I_\rho^{-L} \to \Sigma. \tag{2.6}$$

The construction of Ψ relies on Whitney's theorem; indeed, once considered an embedding $\Omega : \Sigma \to \mathbb{R}^m$, it is possible to define in $\mathbb{R}^m$ the barycenter of $\frac{e^u}{\int_\Sigma e^u dV_g}$ (always for $u \in I_\rho^{-L}$) as $\widetilde{\Psi}(u) := \frac{\int_\Sigma \Omega(x) e^u dV_g(x)}{\int_\Sigma e^u dV_g}$. Moreover, when $L \gg 0$, since $\widetilde{\Psi}$ turns out to belong to a sufficiently small neighborhood of $\Omega(\Sigma)$, we are able to project $\widetilde{\Psi}(u)$ on $\Omega(\Sigma)$ (taking the closest point for example). Thus, just coming back via Ω, we get Ψ which turns out to be continuous (see [29] for details).

At last, since the composition of the former map with the latter can be taken to be homotopic to the identity on Σ, the following result holds true.

Proposition 2.4. *If $\rho \in (8\pi, 16\pi)$, there exists $L > 0$ such that I_ρ^{-L} is not contractible.*

Remark 2.5. As mentioned before Lemma 2.2, when $l > 1$, the constant in (2.2) can be taken arbitrarily close to $\frac{1}{(l+1)16\pi}$.

In [16] and [29], analogously to Lemma 2.3, it is proved that when $\rho \in (8k\pi, 8(k+1)\pi)$ and $I_\rho(u)$ is very negative, the probability measure $\frac{e^u}{\int_\Sigma e^u dV_g}$ concentrates near at most k points of Σ.

From these arguments, when $\rho \in (8k\pi, 8(k+1)\pi)$, we are led to consider the family Σ_k of *formal barycenters of Σ of order k*, which is the space of probability measures whose supports are distributed in at most k points of Σ. By means of Σ_k, it is still possible to prove that there exists $L > 0$ such that I_ρ^{-L} is not contractible. The proof can be found in the aforementioned articles and this case will be considered in a future paper.

2.2. *A Deformation Lemma and a Compactness Result*

It is well known that, if $I \in C^1(\overline{H}^1(\Sigma), \mathbb{R})$ satisfies the Palais–Smale condition, a classical deformation lemma ensures that we have the following alternative: either

1. I^a is a deformation retract of I^b $(a < b)$, or
2. there is a critical point $\bar{u}$ for the functional I, with $a \leq I(\bar{u}) \leq b$.

This lemma, which is usually employed to derive existence of critical points, can be obtained by considering the pseudo-gradient vector field associated to I.

Unfortunately, for our functional, I_ρ, the (PS)-condition is known to hold only for bounded sequences; Lucia in [26] overcomes this problem modifying the usual flow. We present his result, combined with Theorem 4 of [13].

Lemma 2.6. *Consider $c \in \mathbb{R}$ and let $U \subset \overline{H}^1(\Sigma)$ be an open neighborhood of Z_c, possibly empty. The following alternative holds: either*

1. *$\exists \delta > 0$ such that $I_\rho^{c+\delta} \setminus U$ can be deformed in $I_\rho^{c-\delta}$ in a way that $I_\rho^{c-2\delta} \setminus U$ holds steady, or*
2. *for any $\delta > 0$ there exists $\rho_n \to \rho$, $\rho_n \leq \rho$, such that I_{ρ_n} admits a critical point $u_n \in \overline{H}^1(\Sigma) \setminus U$ and $c - \delta \leq I_\rho(u_n) \leq c + \delta$.*

By deformation retract onto $A \subset X$ we mean a continuous map $\eta : [0, 1] \times X \to X$ such that $\eta(t, u_0) = u_0$ for every $(t, u_0) \in [0, 1] \times A$ and such that $\eta(1, \cdot)_{|B}$ is contained in A.

To prove the lemma, one argues as follows: assuming the second alternative false, let $\tau > 0$ be such that $I_{\tilde{\rho}}$ has no critical point $\bar{u} \in \overline{H}^1(\Sigma)\setminus U$ for $\tilde{\rho} \in (\rho - \tau, \rho)$, with $I_{\tilde{\rho}}(\bar{u}) \in [a, b]$. The strategy of the proof consists in constructing, under these hypotheses, a flow which deforms $I_\rho^b \setminus U$ onto a subset of I_ρ^a by keeping bounded every integral curve (with bounds depending on the initial datum, a, b and τ). To do this let Z be defined by:

$$Z(u) := -[|\nabla J(u)|\nabla I_\rho(u) + |\nabla I_\rho(u)|\nabla J(u)], \tag{2.7}$$

where $J(u) = -\log \int_\Sigma h(x) e^u dV_g$, $u \in \overline{H}^1(\Sigma)$.

Then choose a smooth non-decreasing cut-off function $\omega_\tau : \mathbb{R} \to [0, 1]$ satisfying

$$0 \le \omega_\tau \le 1, \quad \omega_\tau(\zeta) = 0 \ \forall \zeta \le \tau, \quad \omega_\tau(\zeta) = 1 \ \forall \zeta \ge 2\tau,$$

and consider the local flow $\eta = \eta(t, u_0)$ defined by the Cauchy problem:

$$\frac{d_x u}{d_x t} = -\omega_\tau\left(\frac{|\nabla I_\rho(u)|}{|\nabla J(u)|}\right)\nabla I_\rho(u) + Z(u), \quad u(0) = u_0, \tag{2.8}$$

where $\omega_\tau\big(\frac{|\nabla I_\rho(u)|}{|\nabla J(u)}\big)$ is understood to be equal to 1 when $\nabla J(u) = 0$. A key point is to notice that $\langle Z(u), \nabla I_\rho(u)\rangle \le 0$, and that if $\langle Z(u_k), \nabla I_\rho(u_k)\rangle$ tends to zero along some sequence $(u_k)_k$, then $\lim_{k\to\infty} \frac{Z(u_k)}{|\nabla J(u_k)|} = 0$.

This lemma is still too weak because it only guarantees that if sublevels are not homotopically equivalent, then there exists a sequence of solutions of perturbed problems. Nevertheless, if $\rho \ne 8k\pi$, as in our case, a compactness result due to Li [22], comes to our rescue.

Theorem 2.7. *If $\rho \ne 8k\pi$, $k \in \mathbb{N}$, $\rho_n \to \rho$ and $(u_n)_n \subset H^1(\Sigma)$ is a sequence of solutions of* (1.1) *relative to ρ_n such that $\int_\Sigma h e^u dV_g = 1$, then $(u_n)_n$ admits a subsequence converging in C^2 to a solution of* (1.1) *relative to ρ.*

To establish this result it is crucial a theorem of Brezis and Merle [4], and its completion given by Li and Shafrir [23], concerning the blow up of solutions to

$$-\triangle w_n = V_n(x) e^{w_n} \quad \text{on } \Omega \subset \mathbb{R}^2.$$

In particular in [23] it is proved that in case of blow up

$$V_n e^{w_n} \rightharpoonup \sum_{i=1}^{m} 8\pi m_i \delta_{x_i},$$

where $m_i \in \mathbb{N}$ and $x_i \in \Omega$. A similar result holds for compact surfaces and moreover in [22] it is shown that $m_i = 1$ for any i. From these considerations Theorem 2.7 follows immediately.

So, employing together Lemma 2.6 and Theorem 2.7 (just considering the right normalization), it is immediate to establish a strong result concerning our functional I_ρ, through and through analogous to the classical aforementioned deformation lemma.

Corollary 2.8. *If $\rho \neq 8k\pi$, $c \in \mathbb{R}$ and U is an open neighborhood of Z_c, then $\exists \delta > 0$ such that $I_\rho^{c+\delta} \setminus U$ can be deformed into $I_\rho^{c-\delta}$ in a way that $I_\rho^{c-2\delta} \setminus U$ holds steady.*

In [28] (see also [16]), Corollary 2.8 is used to prove that, since I_ρ stays uniformly bounded on the solutions of (1.1) (by Theorem 2.7), it is possible to retract the whole Hilbert space $\overline{H}^1(\Sigma)$ onto a high sublevel I_ρ^b, $b \gg 0$. More precisely:

Proposition 2.9. *If $\rho \in (8k\pi, 8(k+1)\pi)$ for some $k \in \mathbb{N}$ and if b is sufficiently large positive, the sublevel I_ρ^b is a deformation retract of $\overline{H}^1(\Sigma)$ and hence is contractible.*

Finally, Corollary 2.8, Proposition 2.9 and the non contractibility of I_ρ^a, $a \ll 0$, see Remark 2.5, allow to establish a general existence result.

Theorem 2.10. *If $\rho \in (8k\pi, 8(k+1)\pi)$, there exists a solution of* (1.1).

A complete proof of previous theorem can be found in [16], but there the approach is quite different.

2.3. *Lusternik–Schnirelman Category*

We recall first the definition of Lusternik–Schnirelman category (category, for short); then, following [18], we will introduce the more general notion of (Lusternik–Schnirelman) relative category and state some of its elementary properties. We will see that this is a powerful tool in critical point theory to obtain multiplicity results.

Let X be a topological space and A a subset of X. The *category of A with respect to X*, denoted by $\mathrm{cat}_X A$, is the least integer k such that $A \subset A_1 \cup \cdots \cup A_k$, with A_i $(i = 1, \ldots, k)$ closed and contractible in X. We set $\mathrm{cat}_X \emptyset = 0$ and $\mathrm{cat}_X A = +\infty$ if there are no integers satisfying the demand.

Now let X be a topological space and Y a closed subset of X. A closed subset A of X *is of the kth category relative to Y* (we write $\mathrm{cat}_{X,Y} A = k$) if k is the least positive integer such that there exist $A_i \subset A$ closed and $h_i : A_i \times [0,1] \to X$, $i = 0, \ldots, k$, such that

(i) $A = \bigcup_{i=0}^k A_i$
(ii) $h_i(x, 0) = x \ \forall i \ \forall x \in A_i$
(iii) $\forall i \geq 1$
  (a) $\exists x_i \in X$ such that $h_i(x, 1) = x_i$
  (b) $h_i(A_i \times [0,1]) \cap Y = \emptyset$
(iv) $i = 0$
  (a) $h_0(x, 1) \in Y \ \forall x \in A_0$
  (b) $h_0(y, t) = y \ \forall y \in Y \ \forall t \in [0,1]$.

If one such k does not exist, then we set $\mathrm{cat}_{X,Y} A = \infty$.

Starting from the above definition, it is easy to check that the following properties hold true.

Proposition 2.11. *Let A, B and Y be closed subsets of X:*

1. *if $Y = \emptyset$, then $A_0 = \emptyset$ and* $\mathrm{cat}_{X,\emptyset} = \mathrm{cat}_X A$;
2. *if $A \subset B$, then* $\mathrm{cat}_{X,Y} A \leq \mathrm{cat}_{X,Y} B$;

3. *if there exists an homeomorphism* $\phi : X \to X'$ *such that* $Y' = \phi(Y)$ *and* $A' = \phi(A)$, *then* $\mathrm{cat}_{X',Y'}A' = \mathrm{cat}_{X,Y}A$;
4. *if* $X' \supset X \supset A$ *and* $r : X' \to X$ *is a retraction such that* $r^{-1}(Y) = Y$ *and* $r^{-1}(A) \supset A$, *then* $\mathrm{cat}_{X',Y}A \geq \mathrm{cat}_{X,Y}A$.

Usually, the notion of category is employed to find critical points of a functional I on a manifold X, in connection with the topological structure of X. Moreover a classical theorem by Lusternik–Schnirelman shows that either there are at least $\mathrm{cat}_X X$ critical points of I on X, or at some critical level of I there is a continuum of critical points (see, for example, [3]).

This result cannot directly help us because, since we look for critical points on $\overline{H}^1(T)$, we would take $X = \overline{H}^1(T)$ which, clearly, has category equal to 1 (being contractible).

So we will need a generalization of such a theorem which involves relative category of sublevels.

Theorem 2.12 ([18]). *Let* H *be a Hilbert space and* $I \in C^1(H, \mathbb{R})$ *a functional satisfying the* (PS)*-condition. If* $-\infty < a < b < +\infty$ *and* $Z_a = Z_b = \emptyset$, *then*

$$\#\{\, u \in I^{-1}([a, b]) \mid \nabla I(u) = 0 \,\} \geq \mathrm{cat}_{I^b, I^a}(I^b).$$

3. Proof of Theorem 1.2

When Σ is a flat torus with fundamental cell domain $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[\frac{1}{2}, \frac{1}{2}\right]$, $h \equiv 1$ and $\rho \in (8\pi, 4\pi^2)$, $u = 0$ turns out to be a strict local minimum for I_ρ. To show it, Struwe and Tarantello [33], just observed that the second variation of I_ρ at $u = 0$ in direction $v \in \overline{H}^1(T)$ can be estimated as follows

$$D^2 I_\rho(0)[v, v] = \|v\|^2 - \rho \int_\Sigma v^2 dx \geq \left(1 - \frac{\rho}{4\pi^2}\right)\|v\|^2. \tag{3.1}$$

This feature gives the functional a mountain pass geometry and permitted to exhibit a saddle point of I_ρ, see Theorem 1.1. Moreover, since h is constant, if u is a solution of (1.3), the functions $u_{x_0}(x) := u(x - x_0)$ still solve (1.3), for any $x_0 \in T$; so from Theorem 1.1 we can deduce the existence of an infinite number of solutions of (1.3).

Now, we want to investigate what happens perturbing h and g. The same procedure of [33] ensures the presence of a strict local minimum close to $u = 0$ and of a saddle point of

$$I_\rho(u) = \frac{1}{2}\int_T |\nabla_g u|^2 dV_g - \rho \log \int_T h e^u dV_g. \tag{3.2}$$

However, if u is a non-trivial critical point we cannot guarantee criticality of the translated functions u_{x_0} anymore.

3.1. *Definition of a New Functional*

Let us point out that, if h and g are respectively sufficiently close to 1 and dx^2, there exist $r > 0$ and a strict local minimum, $\bar{u}$, in $B_r(0)$ such that $M := \inf I_{\rho|\partial B_r(0)} >$

$I_\rho(\bar{u}) := m$ and that $D^2 I_{\rho|B_r(0)}$ is positive definite. The proof below will actually rely only on the existence of a local minimum, which is the requirement of Remark 1.3.

Let us consider an increasing cut-off function

$$\zeta : \mathbb{R} \to \mathbb{R} \text{ such that } \zeta(x) < -L \quad \text{if } x < \frac{I_\rho(0) + M}{2} \quad \text{and} \quad \zeta(x) = x \quad \text{if } |x| \geq M,$$

where L is a large positive constant to be fixed.

By means of ζ we are able to construct a new functional $\widetilde{I}_\rho$ which coincides with I_ρ out of a neighborhood of $\bar{u}$ and which assumes large negative values near $\bar{u}$:

$$\widetilde{I}_\rho(u) := \begin{cases} I_\rho(u) & \text{if } u \in \overline{H}^1(T) \backslash B_r(0) \\ \zeta(I_\rho(u)) & \text{if } u \in B_r(0). \end{cases}$$

The choice of $\widetilde{I}_\rho$, instead of I_ρ, is convenient because of the greater topological complexity of its low sublevels; in particular we will use that they are disconnected (just for the presence of a strict local minimum).

Finally, let us remark that saddle points of $\widetilde{I}_\rho$ are also saddle points of I_ρ, hence we can limit ourselves to study $\widetilde{I}_\rho$.

3.2. *Estimate by Means of Category*

Let X denote the contractible cone over T and let Y be its boundary; they can be represented as

$$X = \frac{T \times [0, 1]}{T \times \{0\}}, \quad Y = \frac{T \times (\{0\} \cup \{1\})}{T \times \{0\}}. \tag{3.3}$$

Once $\rho \in (8\pi, 4\pi^2)$ is fixed, to prove Theorem 1.2 it is sufficient to establish the following chain of inequalities:

$$\begin{aligned} \#\{\text{critical points of } \widetilde{I}_\rho \text{ in } -L \leq \widetilde{I}_\rho \leq b\} &\overset{1}{\geq} \operatorname{cat}_{\widetilde{I}^b_\rho, \widetilde{I}^{-L}_\rho} \widetilde{I}^b_\rho \overset{2}{\geq} \operatorname{cat}_{\widetilde{I}^b_\rho, \phi(Y)} \widetilde{I}^b_\rho \\ &\overset{3}{\geq} \operatorname{cat}_{\widetilde{I}^b_\rho, \phi(Y)} \phi(X) \overset{4}{\geq} \operatorname{cat}_{\phi(X), \phi(Y)} \phi(X) \\ &\overset{5}{\geq} \operatorname{cat}_{X,Y} X \overset{6}{\geq} 2, \end{aligned} \tag{3.4}$$

where ϕ is the homeomorphism on the image defined as follows:

$$\begin{aligned} \phi : X &\longrightarrow \overline{H}^1(T) \\ (x, t) &\longmapsto t\,\varphi_{\lambda,x}, \end{aligned}$$

with L, λ e b suitable constants, clearly depending on ρ, which will be fixed further on.

The main idea is to estimate the number of critical points in $\{-L \leq \widetilde{I}_\rho \leq b\}$ by the category of $\widetilde{I}^b_\rho$ relative to $\widetilde{I}^{-L}_\rho$ applying Theorem 2.12 (pretending for the moment that all the hypotheses are fulfilled), then to find in $\widetilde{I}^b_\rho$ a set homeomorphic

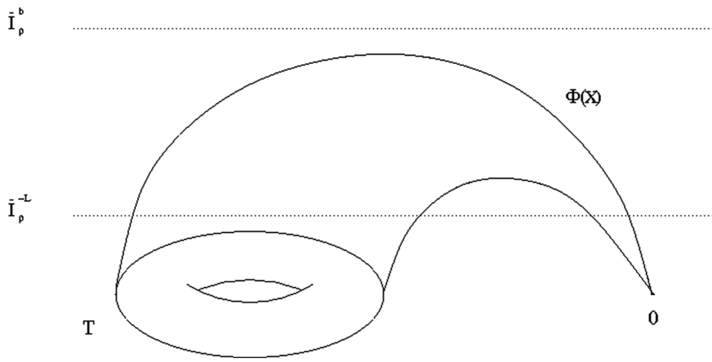

Figure 1. Values attained by $\widetilde{I}_\rho$ on $\phi(X)$.

to the topological cone X, whose boundary is contained in $\widetilde{I}_\rho^{-L}$ (see Figure 1). Next, proving that there exists a positive constant $\tilde{C}_{\rho,b}$ depending only on ρ and b, such that it is possible to extend Ψ to $I_\rho^b \backslash \overline{B}_{\tilde{C}_{\rho,b}}$, we define a projection of $\widetilde{I}_\rho^b (= I_\rho^b)$ onto $\phi(X)$ (see Subsection 3.3, point 4.). Finally the fifth inequality, which exploits the homeomorphism ϕ, allows us to reduce the problem to the study of the category of a finite dimensional space.

Following the notation of [29] we want to show how to extend Ψ, introduced in (2.6), to all u such that, given an ε opportunely fixed, there exists a point $p \in T$ where the function e^u concentrates, namely

$$\frac{\int_{T \backslash B_r(p)} e^u dV_g}{\int_T e^u dV_g} < \varepsilon.$$

In particular we claim that:

$$\text{given } b \in \mathbb{R}, \text{ there exists } \widetilde{C}_{\rho,b} \text{ such that it is possible to define } \Psi : I_\rho^b \backslash \overline{B}_{\tilde{C}_{\rho,b}} \to \Sigma. \tag{3.5}$$

By previous consideration, we only need to prove that if e^u does not concentrate, then $\|u\|$ is bounded by a constant depending only on ρ and b. To do this we recall a lemma, proved in [29].

Lemma 3.1. *Let ε and r be positive numbers. Suppose that for a non-negative function $f \in L^1(T)$ with $\|f\|_{L^1(T)} = 1$ there holds*

$$\int_{B_r(p)} f dV_g < 1 - \varepsilon \quad \textit{for any } p \in T.$$

Then there exists $\bar{\varepsilon} > 0$ and $\bar{r} > 0$, depending only on ε and r (but not on f), and 2 points $\bar{p}_1, \bar{p}_2 \in T$ (which depend on f) satisfying

$$\int_{B_{\bar{r}}(\bar{p}_1)} f \, dV_g \geq \bar{\varepsilon}, \quad \int_{B_{\bar{r}}(\bar{p}_2)} f \, dV_g \geq \bar{\varepsilon}; \quad B_{2\bar{r}}(\bar{p}_1) \cap B_{2\bar{r}}(\bar{p}_2) = \emptyset.$$

Let $u \in I_\rho^b$ such that for any $p \in T$

$$\frac{\int_{T\setminus B_r(p)} e^u dV_g}{\int_T e^u dV_g} \geq \varepsilon;$$

then Lemma 2.3 implies the existence of two positive numbers $\bar{\varepsilon}$ and $\bar{r}$ (independent of u) and two points $\bar{p}_1$ and $\bar{p}_2$ (which instead depend on u) such that

$$\frac{\int_{B_{\bar{r}}(\bar{p}_i)} e^u dV_g}{\int_T e^u dV_g} \geq \bar{\varepsilon} \quad \text{for } i = 1, 2 \quad \text{and} \quad B_{2\bar{r}}(\bar{p}_1) \cap B_{2\bar{r}}(\bar{p}_2) = \emptyset.$$

So we can apply Lemma 2.2 with $\delta_0 = 2\bar{r}$, $\Omega_i = B_{\bar{r}}(\bar{p}_i)$ and $\gamma_0 = \min[\bar{\varepsilon}, \frac{1}{3}]$; in particular, choosing $\tilde{\varepsilon}$ such that $\frac{4\pi^2}{32\pi-\tilde{\varepsilon}} < \frac{1}{2}$, we obtain the existence of a constant $K = K(\varepsilon, r)$ such that, for any $u \in \overline{H}^1(T)$,

$$\log \int_T e^u dx \leq K + \frac{1}{32\pi - \tilde{\varepsilon}} \int_T |\nabla u|^2 dx.$$

Then

$$b \geq I_\rho(u) \geq \frac{1}{2}\int_T |\nabla u|^2 dx - \rho K - \frac{\rho}{32\pi - \tilde{\varepsilon}} \int_T |\nabla u|^2 dx \geq a\|u\|^2 - \rho K,$$

where $a = \frac{1}{2} - \frac{\rho}{32\pi-\tilde{\varepsilon}} > 0$. At last, as K does not depend on u, the claim is proved.

Let us fix some constants: we choose L in such a way that, for any λ large enough such that $T_\lambda \subset \widetilde{I}_\rho^{-L}$, the map $x \mapsto \Psi(\varphi_{\lambda,x})$ is homotopic to the identity; then we fix b sufficiently large so that $\phi(X) \subset \widetilde{I}_\rho^b$, I_ρ^b is a deformation retract of $\overline{H}^1(T)$ (Proposition 2.9) and so as to have $\max I_{\rho|\partial B_{2r}(0)} \leq b$, where r is as in Subsection 3.1; by this choice $\widetilde{I}_\rho^b = I_\rho^b$. Moreover without any loss of generality we can assume that neither $-L$ nor b are critical levels. In the end we take λ such that following conditions are both verified: $T_\lambda \subset \widetilde{I}^{-L}$ and $\min_{x\in T} \|\varphi_{\lambda,x}\| > \widetilde{C}_{\rho,b}$, where $\widetilde{C}_{\rho,b}$ is the constant determined in the previous claim.

3.3. *Proof of the Inequalities in* (3.4)

To get Theorem 1.2 it remains only to prove the inequalities in (3.4).

1. $$\#\{\text{critical points of } \widetilde{I}_\rho \text{ in } \{-L \leq \widetilde{I}_\rho \leq b\}\} \geq \operatorname{cat}_{\widetilde{I}_\rho^b, \widetilde{I}_\rho^{-L}} \widetilde{I}_\rho^b$$

As we do not know if the (PS)-condition is satisfied, we cannot appeal to Theorem 2.12; however it is crucial to remark that, in the proof of the aforementioned theorem ([18]), the (PS)-condition is used only twice to apply the classical Deformation Lemma (see for example [3] or [13]). It is easy to see that both times Corollary 2.8 comes to our rescue. The only important thing to remark is that in the neighborhood of the origin, where $\widetilde{I}_\rho$ differs from I_ρ (and so we cannot apply the corollary), we can deform along the flux generated by a cutoff of the opposite of the gradient.

2. $$\mathrm{cat}_{\widetilde{I}^b_\rho,\widetilde{I}^{-L}_\rho}\widetilde{I}^b_\rho \geq \mathrm{cat}_{\widetilde{I}^b_\rho,\phi(Y)}\widetilde{I}^b_\rho$$

It is worth pointing out that in the hypotheses of Theorem 1.2 (merely when g is sufficiently close to dx^2 and h to 1), the map Ψ introduced in (2.6) turns out to be a diffeomorphism between T_λ and Σ. So we can define a diffeomorphism $\omega : \Sigma \to \Sigma$ such that $\omega(\Psi(\varphi_{\lambda,x})) = x$.

Next, reminding that $\widetilde{I}^{-L}_\rho$ is the disjoint union of I^{-L}_ρ and a neighborhood U of the origin, we can consider the following map:

$$\begin{aligned}\chi : \widetilde{I}^{-L}_\rho &\longrightarrow \phi(Y)\\ u &\longmapsto \varphi_{\lambda,\omega(\Psi(u))} \quad u \in I^{-L}_\rho\\ u &\longmapsto 0 \qquad\qquad u \in U.\end{aligned}$$

Now, our purpose is to find a deformation retract $\big(\text{in } \widetilde{I}^b_\rho\big)$ of $\widetilde{I}^{-L}_\rho$ onto $\phi(Y)$. First of all, let us set

$$\begin{aligned}\gamma : \widetilde{I}^{-L}_\rho \times [0,1] &\longrightarrow \overline{H}^1(T)\\ (u,t) &\longmapsto (1-t)u + t\chi(u).\end{aligned}$$

Then, thanks to our choice of b we know that $\widetilde{I}^b_\rho \equiv I^b_\rho$ is a deformation retract of $\overline{H}^1(T)$, namely there exists a continuous map $\tau : \overline{H}^1(T) \to \widetilde{I}^b_\rho$ such that $\tau_{|\widetilde{I}^b_\rho} = \mathrm{Id}_{|\widetilde{I}^b_\rho}$. So composing γ and τ we get the map $\big(h := \tau \circ \gamma : \widetilde{I}^{-L}_\rho \times [0,1] \to \widetilde{I}^b_\rho\big)$ we were looking for. Indeed, for any $u \in \widetilde{I}^{-L}_\rho$, $h(u,0) = u$ and $h(u,1) = \chi(u) \in \phi(Y)$, while, for any $(y,t) \in \phi(Y) \times [0,1]$, $h(y,t) = y$ (being $\chi_{|\phi(Y)} = \mathrm{Id}_{|\phi(Y)}$).

At last, if A_i and h_i $\big(i = 1, \ldots, \mathrm{cat}_{\widetilde{I}^b_\rho,\widetilde{I}^{-L}_\rho}\widetilde{I}^b_\rho\big)$ fulfill the conditions of the definition of relative category for $\mathrm{cat}_{\widetilde{I}^b_\rho,\widetilde{I}^{-L}_\rho}$, it is easy to prove that A_0, $h * h_0$ and A_i, h_i $(i \geq 1)$ verify the definition of category for $\mathrm{cat}_{\widetilde{I}^b_\rho,\phi(Y)}\widetilde{I}^b_\rho$, where $h * h_0 : A_0 \times [0,1] \to \widetilde{I}^b_\rho$ is defined as follows:

$$h * h_0(x,t) := \begin{cases} h(h_0(x,2t),0) & t \leq \dfrac{1}{2}\\[2mm] h(h_0(x,1),2t-1) & t \geq \dfrac{1}{2}.\end{cases}$$

3. $$\mathrm{cat}_{\widetilde{I}^b_\rho,\phi(Y)}\widetilde{I}^b_\rho \geq \mathrm{cat}_{\widetilde{I}^b_\rho,\phi(Y)}\phi(X)$$

It is a merely map of the Point 2 of Proposition 2.11.

4. $$\mathrm{cat}_{\widetilde{I}^b_\rho,\phi(Y)}\phi(X) \geq \mathrm{cat}_{\phi(X),\phi(Y)}\phi(X)$$

Thanks to Proposition 2.11, Point 4, if we construct a continuous map $r : \widetilde{I}^b_\rho \to \phi(X)$ such that $r_{|\phi(X)} = \mathrm{Id}_{|\phi(X)}$ and that $r^{-1}(\phi(Y)) = \phi(Y)$, we are done. Let us define

$C_{\rho,b} := \min_{x\in T} \|\varphi_{\lambda,x}\|$, which is bigger than $\widetilde{C}_{\rho,b}$, according to our choice of λ; then we are able to define Ψ and also χ on the set $\{v \in \overline{H}^1(T) : \|v\| \geq C_{\rho,b}\}$.

Therefore the following map is well defined:

$$\begin{aligned} r : \widetilde{I}_\rho^b &\longrightarrow \phi(X) \\ 0 &\longmapsto 0 \\ u \in \{\|v\| \geq C_{\rho,b}\} &\longmapsto \eta(\mathrm{dist}_{x\in T}(u, \varphi_{\lambda,x}))\,\chi(u) \\ u \in \{\|v\| \leq C_{\rho,b}\} &\longmapsto \frac{\|u\|}{C_{\rho,b}}\, r\left(\frac{C_{\rho,b}}{\|u\|}u\right), \end{aligned}$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth strictly decreasing function, such that $\eta(0) = 1$ and $\eta([\frac{1}{2}, +\infty)) = \frac{1}{3}$.

Finally it is easy to verify that r is continuous and such that $r_{|\phi(X)} = \mathrm{Id}_{|\phi(X)}$ and $r^{-1}(\phi(Y)) = \phi(Y)$.

5. $$\mathrm{cat}_{\phi(X),\phi(Y)}\phi(X) \geq \mathrm{cat}_{X,Y}X$$

We just need to use Point 3 of Proposition 2.11, since ϕ is an homeomorphism on the image.

6. $$\mathrm{cat}_{X,Y}X \geq 2$$

Let consider the closed sets A_i verifying the conditions of the definition of relative category.

First of all, we claim that A_0 is disconnected. This will be enough to guarantee that $\mathrm{cat}_{X,Y}X \geq 1$, because otherwise it would be $A_0 = X$, which is connected.

Let us denote by $Y_0 := T \times \{0\}/T \times \{0\}$ and $Y_1 := T \times \{1\}/T \times \{0\}$ the two disconnected components of Y.

By definition we know that $Y_0 \cup Y_1 = Y \subset A_0$ and that there exists $h_0 : A_0 \times [0, 1] \to X$ continuous with the properties: $h_0(A_0, 1) \subset Y$ and $h_{0|Y\times[0,1]} \equiv \mathrm{Id}_Y$. Now, if A_0 was connected we would get a contradiction because $h_0(A_0, 1)$ would be connected (by continuity of h_0) and disconnected being the union of Y_0 and Y_1.

Thus we can consider the connected component B of A_0 containing Y_1 and its complementary in A_0, $C := A_0 \backslash B$; these sets are both closed, so we can make use of the following lemma.

Lemma 3.2 (Urysohn). *Let X be a finite dimensional vector space; given $B, C \subset X$ closed, there exists $\varphi : X \to \mathbb{R}$, $\varphi \in C^\infty$, such that $\varphi(x) = 1$ for any $x \in B$, $\varphi(x) = 0$ for any $x \in C$ and $\varphi(x) \in (0, 1)$ for any $x \in X\backslash(B \cup C)$.*

Applying Sard Lemma to $f := \varphi_{|T\times[0,1]}$ (where $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is given by Urysohn Lemma), we get the existence of a value $a \in (0, 1)$ such that $\{f = a\}$ is a 2-manifold. We stress that $\{f = a\} \subset X\backslash A_0$ and $\{f = a\}$ disconnects Y_0 and Y_1, being $f_{|Y_0} = 0$ and $f_{|Y_1} = 1$.

Always applying Sard Lemma to

$$g : T \times [0, 1] \longrightarrow S^1$$

$$((x_1, x_2), t) \ \mapsto \ \begin{pmatrix} \cos(2\pi x_1) \\ \sin(2\pi x_1) \end{pmatrix} \tag{3.6}$$

we can deduce that there exists $\bar{\theta} \in S^1$ such that $M := \{(x, t) \in \{f = a\} : g(x, t) = \bar{\theta}\}$ is a 1-manifold; in particular M is the union of a finite number of submanifolds diffeomorphic to S^1: $M = \bigcup_{i=1}^{j} M_i$.

Now, to get our thesis it will be enough to show that at least one M_i is non-contractible in $X \backslash Y$, since $M \subset \{f = a\} \subset X \backslash A_0$. Moreover we can limit ourselves to prove non-contractibility in $\Theta \backslash Y$, where $\Theta = \{(x, t) \in T \times [0, 1] : g(x) = \bar{\theta}\}$.

Using again Urysohn Lemma (with $B_i = M_i$ and $C_i = \Theta \cap Y$) and Sard Lemma, we obtain j 2-manifolds with boundary, $N_i := \{f_i \geq a_i (> 0)\}$; clearly it is possible to choose a_i in such a way that N_i are pairwise disjoint and contractible in Θ. Let denote by W_{i_k}, $k = 1, \ldots, l_i$, the connected components on the boundary of N_i.

Then, there will be a point x such that the segment s connecting $(x, 0)$ and $(x, 1)$ intersects in an even number of points each W_{i_k}. Let us define a path γ. Starting from $(x, 0)$ let us follow s up to its first (possible) intersection with a W_{i_k}; now, the previous arguments ensure that it is possible to follow W_{i_k} up to its intersection with s closest to $(x, 1)$, which by our choice of s is different from the first one. Finally, repeating this procedure a finite number of times, we will reach $(x, 1)$ (see Figure 2).

We got then a contradiction because we constructed a curve γ connecting Y_0 and Y_1 which does not intersect M.

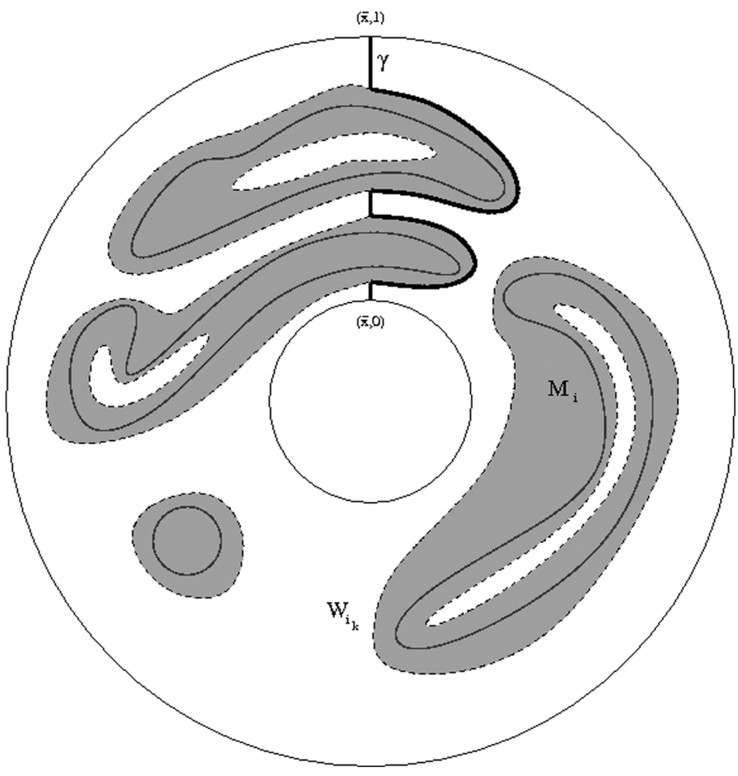

Figure 2. If by contradiction every M_i is contractible in Θ....

Remark 3.3. We believe that, if X, Y are as in (3.3), with T replaced by any compact topological space Z, we should have $\mathrm{cat}_{X,Y}X \geq \mathrm{cat}_Z Z$; so in our case we expect that in **6**. the relative category should be 3. This would lead the existence of a third solution in Theorem 1.2.

To support this argument, one can reason as follows: calling u the critical point found in [33] for $g = ds^2$ and $h = 1$, the family $\{u_{x_0}(\cdot) = u(\cdot - x_0), x_0 \in T\}$ constitutes a torial manifold of solutions to (1.3). If this manifold turned out to be non degenerate, in the sense that $\mathrm{Ker}\, I_\rho''(u_0) = \mathrm{Span}\{\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\}$, (which is generally expectable) then it would be possible to apply the perturbative methods [2] for data close to the constant ones (see also [1]). In particular, Corollary 2.13 in the latter reference guarantees the existence of $\mathrm{cat}_T T$ critical points.

Acknowledgments

The author is grateful to Professor Andrea Malchiodi for helpful discussions and for having proposed her this topic. She is supported by MURST within the PRIN 2006 *Variational Methods and Nonlinear Differential Equations.*

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